Cluster kinetics model of particle separation in vibrated granular media

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We model the Brazil-nut effect (BNE) by hypothesizing that granules form clusters that fragment and aggregate. This provides a heterogeneous medium in which the immersed intruder particle rises (BNE) or sinks (reverse BNE) according to relative convection currents and buoyant and drag forces. A simple relationship proposed for viscous drag in terms of the vibrational intensity and the particle to grain density ratio allows simulation of published experimental data for rise and sink times as functions of particle radius, initial depth of the particle, and particle-grain density ratio. The proposed model correctly describes the experimentally observed maximum in risetime.

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The dynamics of granular materials is a topic of enormous practical and industrial importance as well as a subject of considerable scientific interest. Segregation of particles in a vibrated granular medium is a commonly observed, yet deceptively complex, phenomenon that has attracted serious research efforts [1–3]. The Brazil-nut effect (BNE) is an example illustrating how immersed larger particles separate by rising to the top of a granular medium composed of smaller grains. Generally, particles with a larger size or different density compared to grains in the medium will rise under conditions of low vibrational intensity, but may sink if the bed is fluidized by vigorous agitation [2]. The rise or fall depends on a variety of factors including vibrational frequency and amplitude, vessel geometry, relative particle-grain size and density, particle position in the medium, and air temperature, pressure, and humidity. We use "particle" to denote the intruder in a medium composed of grains. In terms of the frequency ω and the amplitude A, the vessel is vibrated with vertical displacement $A\sin(\omega t)$. The vibrational intensity [4] is measured by the ratio of acceleration amplitude to gravitational acceleration, $\Gamma = A\omega^2/g$. The approach is based on the assumption that particle velocity is influenced by convection currents and buoyant and drag forces. We consider only indirectly effects such as static electrical forces and air pressure gradients.

Distribution kinetics has been successfully used for investigating the dynamics of granular materials [5,6] and has provided a new approach for analyzing granular mixing [6] and densification [5]. In granular mixing by tumbling operations, loose particles slide down the inclined surface and/or particle clusters fragment and fall down the incline [5]. In vibrational settling and compaction, clusters evolve during the densification process according to easily solved differential equations for the moments of the cluster size distribution [6]. Essential to the new approach is the consideration that a granular medium has a heterogeneous, clustered structure unless vigorous agitation thoroughly fluidizes the medium.

Moebius *et al.* [1] and Huerta and Ruiz-Suarez [2] reported detailed observations for granular separation experiments. For the rise time as a function of density ratio, $\rho_r = \rho_p/\rho_g$, they observed a maximum rise time for different initial particle depths. For large Γ, experiments [2,7,8] show that large and/or dense particles sink. This behavior is called the reverse Brazil nut effect (RBNE) [9–11].

For a particle immersed in a fluid, drag and buoyant forces determine the terminal velocity. We propose here an approximate model, developed by considering the particle to be immersed in a granular medium that can aggregate into clusters, yielding a heterogeneous "fluid." We consider convection currents and buoyant and drag forces in writing the equation for particle velocity. All these effects depend on cluster aggregation-fragmentation and grain association-detachment at cluster surfaces. This is similar to the competition between percolation and condensation [12]. Two types of granular packing are considered in developing the granular medium density: free grains amorphously packed at low density and coherent clusters of larger density. Individual grains attaching and detaching from the clusters can explain the transition between them. Similar to a liquid-solid phase transition, the free grains are analogous to the liquid phase and clusters to the crystalline phase.

The current paper begins by presenting the migration velocity of the particle, relative to convective velocity in the granular medium, based on a balance of buoyant and drag forces. Cluster kinetics enters by assuming the viscous drag force is influenced by the heterogeneous nature of the granular medium. Scaled dimensionless variables provide a minimal set of parameters to consider. The resulting equations for particle migration velocity, as well as rise and sink times, are tested by comparison with published experimental observations.

A spherical particle of radius R and density ρ_p is immersed in the background of clusters and free grains of average density ρ. A buoyancy and friction-drag force balance

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in terms of the terminal velocity v relative to the convective velocity v_c can be written as follows:

$$6\pi\mu R(v - v_c) = (4/3)\pi R^3 g(\rho - \rho_p), \tag{1}$$

where g is the gravitational acceleration and μ represents the viscosity of the granular medium. The vertical motion of the vessel walls causes convective motion in the granular material. The granules are all lifted during upward wall movement. During downward wall movement, granules and clusters adjacent to the vertical walls are dragged down by friction, causing a net downward drift in the region near the wall. Aided by inertia during the downward wall movement, grains away from the wall are induced to drift upward or downward depending on the nature of the convective cells. The convection rolls can possibly entrain light or heavy particles and, moderated by inertial effects, move them in a variety of patterns [2,9]. We do not attempt to predict the convective velocity or the migration direction of the particle, but assume a given net convective circulation can contribute to the migration velocity v of the particle as expressed in Eq. (1). Nor do we attempt to describe the physics of displaced air flowing through the granular medium and exerting forces on the particle and the granules [6,8,9].

In the absence of convection (v_c=0), the velocity for either the BNE or RBNE can be cast as the equation [9],

$$\pm v = F\left(\frac{\rho_p}{\rho} - 1\right), \tag{2a}$$

where $F=(2/9)R^2 g\rho/\mu$. The drag force is always opposite to the direction of velocity, so its minus sign applies for the BNE and its plus sign for the RBNE. A plot of v vs ρ_p/ρ is a straight line with slope equal to the negative of the intercept for either BNE or RBNE, as found in experiments by Gutierrez *et al.* [9]. Description of this basic observed behavior is considered necessary for validation of the proposed approach.

Some experimental data are presented in terms of the ratio of particle to grain density, $\rho_r=\rho_p/\rho_g$. The bed density ρ is proportional to the grain density ρ_g through the constant volume ratio, r=grain volume/bed volume, thus we have $\rho = r\rho_g$. For the density difference, we can write $(\rho-\rho_p) = \rho_g(r-\rho_r)$. The maximum hexagonal packing ratio is r =0.74, but vibrational compaction [5] typically gives a smaller value, $0.6<r<0.7$. Based on published experimental observations [1,11], we hypothesize that the frictional retardation of the particle is influenced by the amount of clustering in the granular medium. The viscosity of particulate systems, usually represented as a ratio, can be related to energy dissipation [13],

$$\mu/\mu_o = E/E_o, \tag{2b}$$

where E and E_o are the energy dissipation rates for the clustered and the fluidized media, respectively. Clusters beneath the particle tend to support the particle and prevent downward motion, even for particles more dense than the granular medium [14]. Clusters above, on the other hand, can be fragmented by the inertia of the particle. The ratio of energy dissipation accordingly will depend on the ratio of inertial to fluidizing forces, $\rho_p g/\rho_g A\omega^2=\rho_r/\Gamma$, thus in Eq. (2b), E will be a function of ρ_r/Γ. Fluid mechanical derivations of such functions are often power series [13] with coefficients that depend on the presence of other particles and walls. We approximate the power series (having coefficients of order unity) by the geometric power series $[1+y+y^2+\cdots=1/(1-y)]$ in ρ_r/Γ, and propose the simple relationship,

$$\mu = \mu_o(1 - \alpha_o\rho_r/\Gamma)^{-1}. \tag{3}$$

Equation (3) is admittedly heuristic in that it is based on parallel arguments from classical fluid dynamics for frictional drag on a particle immersed in a fluid. Typical buoyancy occurs when Γ increases sufficiently to cause fluidization, i.e., $\alpha_o\rho_r/\Gamma \ll 1$, and the viscosity takes on the value μ_o, independent of the ratio $\rho_r=\rho_p/\rho_g$. In other words, fluidization overcomes jamming and disrupts stress chains by breaking clusters into free particles. As Γ decreases, $\alpha_o\rho_r/\Gamma$ increases while still less than unity, the granular viscosity increases, and the particles tend to be dragged upward by convection currents. Large and dense particles will rise under this condition.

Solving for particle migration velocity shows that the particle velocity is constant with time and depends on its radius R, the convective velocity v_c, and on relative values of ρ_p and ρ,

$$v = dz/dt = v_c + (2/9)R^2 g(\rho_g/\mu_o)(r - \rho_r)(1 - (\alpha_o/\Gamma)\rho_r). \tag{4}$$

We may define dimensionless quantities (scaled by the bed depth h and a reference time h/v_o where $v_o=\mu_o/h\rho_g$) as follows:

$$\theta = tv_o/h, \quad x = z/h, \quad x_0 = z_0/h, \quad \xi = R/h,$$

$$\theta_{\text{rise}} = t_{\text{rise}}v_o/h, \quad \theta_{\text{sink}} = t_{\text{sink}}v_o/h,$$

$$\phi = hg/v_o^2, \quad \eta = v_c/v_o = v_c h\rho_g/\mu_o. \tag{5}$$

Two of the dimensionless parameters have fundamental fluid mechanical significance, the (convection) Reynolds number η and the Froud number ϕ. The scaled governing equation is

$$dx/d\theta = \pm\eta + (2/9)\phi\xi^2(r - \rho_r)(1 - (\alpha_o/\Gamma)\rho_r), \tag{6}$$

where we have explicitly denoted the possibility of upward or downward convection by $\pm\eta$. Clearly $dx/d\theta$ can be positive or negative, denoting upward or downward particle migration, depending on the relative magnitudes of the parameters in Eq. (6). This range of possible migration velocities can explain the variety of observed particle behavior in granular media.

Rise and sink times can be formulated by integrating Eq. (6) from $x=x_0$ to $x=1$ or 0 depending on whether the particle is rising or falling, respectively,

$$\theta_{\text{rise}} = (1 - x_0)\{\eta + (2/9)\phi\xi^2(r - \rho_r)(1 - (\alpha_o/\Gamma)\rho_r)\}^{-1}, \tag{7a}$$

$$\theta_{\text{sink}} = x_0\{-\eta - (2/9)\phi\xi^2(r - \rho_r)(1 - (\alpha_o/\Gamma)\rho_r)\}^{-1}. \tag{7b}$$

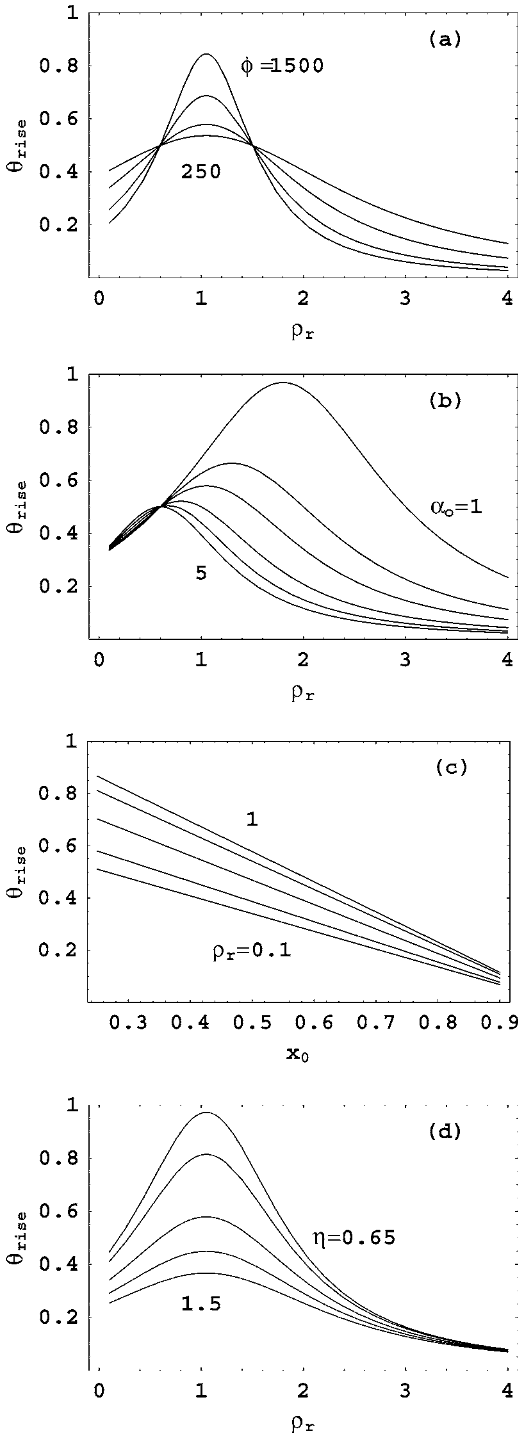

FIG. 1. Risetime vs particle-grain density ratio ρ_r for the clustered (nonfluidized) granular medium: effect of (a) ϕ (=250, 500, 1000, and 1500); (b) α_o (=1, 1.5, 2, 3, 4, and 5); (c) the initial depth x_0 with ρ_r=0.1, 0.25, 0.5, 0.75, and 1; (d) η=0.65, 0.75, 1, 1.25, and 1.5. Unless otherwise specified, the other parameters are R =20 mm, Γ=3, r=0.6, α_o=2, ϕ=500, x_0=0.5, and η=1.

For a fluidized medium, Γ is so large that $(\alpha_o/\Gamma)\rho_r \ll 1$,

$$\theta_{\text{rise}} = (1 - x_0)\{\eta + (2/9)\phi\xi^2(r - \rho_r)\}^{-1}, \tag{8a}$$

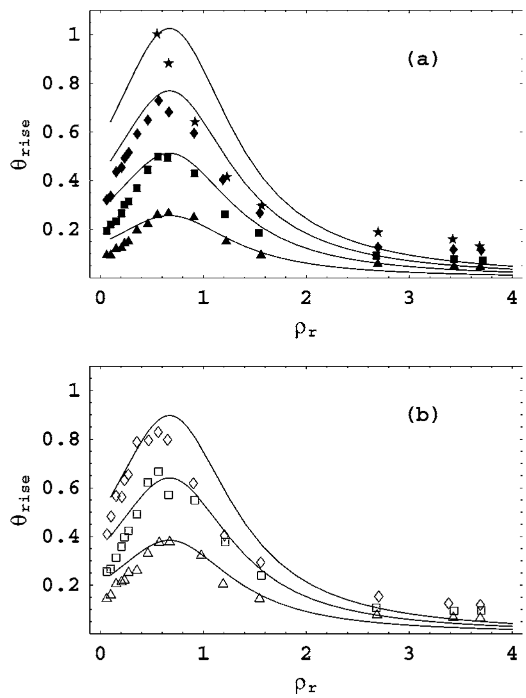

FIG. 2. Comparison of the model [Eqs. (7a) and (7b)] with experimental data [2] for risetimes of varying initial depths x_0 in the clustered granular medium of tapioca spheres. In Eq. (8a), α_o is determined by the position of the maximum, the fitting parameter is ϕ=500 with η=0.75, and the given values are h=21 cm, R =20 mm, Γ=3, and r=0.6. Legend: ▲=5 cm; △=7 cm; ■=9 cm; □=11 cm; ◆=13 cm; ◇=15 cm; ∗=17 cm.

$$\theta_{\text{sink}} = x_0\{- \eta - (2/9)\phi\xi^2(r - \rho_r)\}^{-1}. \tag{8b}$$

Further, when fluidization totally disrupts convection [1], we can set $\eta \approx 0$; then Eqs. (8a) and (8b) are consistent with Eq. (2). Clearly r is the critical ratio [7] for determining whether the particle rises or sinks. This value was found by Yan *et al.* [7] to be between 0.6 and 0.7, consistent with other reported values [3]. If the rise (sink) time starts when the particle is at the bottom (top) of the granular bed, then $x_0=0$ ($x_0=1$). For the totally fluidized bed, only one parameter, ϕ, is available to fit data for rise and sink times.

Huerta and Ruiz-Suarez [2], following Moebius *et al.* [1], recently reported data for large spherical particles (4 cm diameter) of different densities in a column of small seeds (3.1 mm diameter and 0.57 g/cm^3 density). A major result was that the risetime reached a maximum (velocity reached a minimum) at a density ratio of the particle and the grains, ρ_r, somewhat greater than 0.5. Not unexpectedly, the rise time increased with the depth of the particle initially. Experiments [2] showed that larger R led to larger rise velocity and shorter rise times, as reflected by Eqs. (7a) and (7b) for θ_{rise} and θ_{sink}. It was also found that θ_{rise} and θ_{sink} vs ρ_r data for different starting depths collapsed to single curves when normalized by a specified rise or sink time. Equations (7a) and (7b) correspondingly yield single curves because the starting position x_0 cancels with the normalization.

Computations demonstrate that choosing either r=0.6 or 0.7 has negligible effect on values for the rise and sink times.

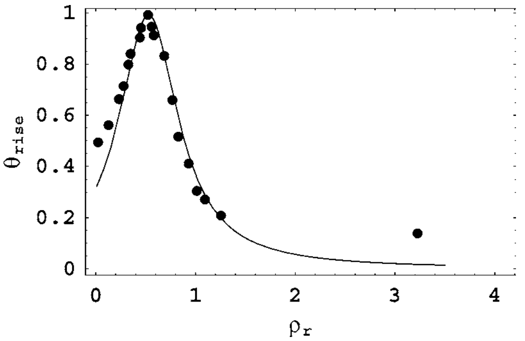

FIG. 3. Comparison of the model [Eqs. (7a) and (7b)] with experimental data [1] for risetimes of vibrated media of glass spheres.

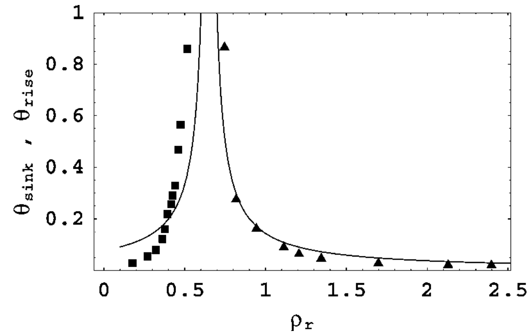

FIG. 4. Comparison of the model [Eqs. (8a) and (8b)] with experimental data [7] for rise ▲ ($\rho_r > r$) and sink ■ ($\rho_r < r$) times in the fluidized granular medium of glass beads.

Thus, given x_0, r, and ρ_r, there are three remaining parameters in the model, η, α_o, and ϕ. Figures 1(a)–1(d) show the effects of parameters on the variation of θ_{rise} with ρ_r. Figure 1(a) shows that θ_{rise} increases and the curve becomes sharper with increasing ϕ. Figure 1(b) shows that α_o plays a crucial role in determining the value of ρ_r at which θ_{rise} is maximum. Figure 1(c) shows the linear variation of θ_{rise} with x_0 for different values of ρ_r, similar to experimental findings [2]. Figure 1(d) shows how the characteristic velocity influences θ_{rise}.

Figure 2 shows the comparison of the model [Eq. (8a)] with experimental data [2] for risetimes of varying initial depths x_0 in the clustered granular medium of tapioca monodisperse spheres. When normalized by dividing by the greatest risetime, the values of risetimes are in the interval [0,1]. The aim was to select one set of parameters for all the data, although choosing different values of α_o and ϕ for the separate curves would yield even better fits. Given R=20 mm, Γ=3, and r=0.6, a maximum value of θ_{rise} for the different initial depths x_0 occurs when $d\theta_{rise}/d\rho_r$=0, thus when

$$\rho_r = (\alpha_o r + \Gamma)/2\alpha_o. \tag{9}$$

Given r and Γ, Eq. (9) allows one to determine α_o from experimental data for the value of ρ_r (=0.67 for data in Fig. 1 of [1]) at which θ_{rise} is maximum. If the characteristic velocity v_o in Eq. (5) is chosen equal to 0.75 v_c, i.e., η =0.75, then the single remaining parameter ϕ is determined to be 500. The fit of the model to the data displayed in Fig. 2 is fair for the smaller values of ρ_r, and somewhat underestimates the migration velocity at larger values of ρ_r. The discrepancy between theory and experiment is less for smaller x_0, where the particle has farther to travel. The maximum risetime and its properties, however, are clearly predicted, suggesting that the model captures significant features of the phenomena, notwithstanding the approximate nature of the model and inevitable scatter in the data.

Figure 3 shows the comparison of the model [Eq. (8a)] with experimental data [1] for risetimes of a spherical intruder 2.54 cm diameter at an initial depth of 4.6 cm in the vibrated granular medium consisting of glass spheres. In Eq. (8a), α_o is determined by the position of the maximum and η=0.75 the single remaining fitting parameter is ϕ=600.

Figure 4 shows the comparison of the model with RBNE experimental data [7] for rise and sink times in the fluidized granular medium of glass beads. We assume η=0 in Eqs. (8a) and (8b). Therefore, the single remaining parameter ϕ is determined to be 8000 to fit the data. The model correctly yields the critical density ratio of 0.65, determined by experimental data [7].

According to the concept proposed here, defining a granular medium temperature would be feasible if the vibrational intensity is large and the bed is fluidized. The grains dissociated from the clusters then dominate the granular medium. The changeover from clustered to free grains is similar to a phase transition from caged to gaslike grains [15]. The key parameters, η and ϕ, contain the convective velocity and the granular medium viscosity. At present there is no obvious way to estimate these quantities except through comparison with experimental data. The same is true for α_o, which dictates how the viscosity changes with Γ and ρ_r.

In summary, we have demonstrated that essential features of vibrationally-induced granular separation (BNE and RBNE) can be explained quantitatively by considering the cluster heterogeneity of the granular medium in which the rising or sinking particle is immersed. The hypothesis that cluster dynamics influence the frictional drag on the buoyed particles determines how immersed particles rise or sink according to relative convection currents and buoyant and drag forces. For small vibrational energies, granular clustering increases the frictional (viscous) drag such that large particles, less or more dense than the granular medium, rise with the convection current. The maximum in the rise time for these particles is a striking experimental observation that is rather accurately described by the proposed model. As vibrational frequency and amplitude increase, the granular medium becomes more fluid and complex processes such as air movement may influence rise or sink times. Adjustable parameters, η and ϕ, embodying effects of viscosity and convection velocity, provide a good fit to the rise and sink times as functions of the particle-grain density ratio. The model generally yields a quantitative explanation for effects of vibrational frequency and amplitude, particle radius, initial depth of the particle, and particle-grain density ratio.

[1] M. E. Möbius, B. E. Lauderdale, S. R. Nagel, and H. M. Jaeger, Nature (London) **414**, 270 (2001).
[2] D. A. Huerta and J. C. Ruiz-Suarez, Phys. Rev. Lett. **92**, 114301 (2004); **93**, 069901 (2004).
[3] T. Shinbrot and F. J. Muzzio, Phys. Rev. Lett. **81**, 4365 (1998).
[4] J. B. Knight, C. G. Fandrich, C. N. Lau, H. M. Jaeger, and S. R. Nagel, Phys. Rev. E **51**, 3957 (1995).
[5] B. J. McCoy and G. Madras, Phys. Rev. E **70**, 051311 (2004).
[6] B. J. McCoy and G. Madras, AIChE J. **51**, 406 (2005).
[7] X. Yan, Q. Shi, M. Hou, and K. Lu, Phys. Rev. Lett. **91**, 014302 (2003).
[8] A. P. J. Breu, H.-M. Ensner, C. A. Kruelle, and I. Rehberg, Phys. Rev. Lett. **90**, 014302 (2003).
[9] G. Gutierrez, O. Pozo, L. I. Reyes *et al.*, Europhys. Lett. **67**, 369 (2004).
[10] T. Shinbrot, Nature (London) **429**, 352 (2004).
[11] A. Kudrolli, Rep. Prog. Phys. **67**, 209 (2004).
[12] D. C. Hong, P. V. Quinn, and S. Luding, Phys. Rev. Lett. **86**, 3423 (2001).
[13] J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics with Special Applications to Particulate Media* (Prentice-Hall, Englewood Cliffs, NJ, 1965), Chap. 9.
[14] Y. Nahmad-Molinari, G. Canul-Chay, and J. C. Ruiz-Suarez, Phys. Rev. E **68**, 041301 (2003).
[15] D. C. Hong, P. V. Quinn, and S. Luding, Phys. Rev. Lett. **86**, 3423 (2001).